

Algebraic Number Theory

(Midsem test, BMath-3 and MMath-2, 2026)

Instructions: Total time 3 Hour. Solve any **five** problems for a max score of 30. Use terminologies, notations and results as covered in the course, no need to prove such results.

1. (a) Let K be a number field and $\mathcal{O}_K$ its ring of integers. Let $I \subset \mathcal{O}_K$ be a nonzero ideal. Prove that $I \cap \mathbb{Z} \neq 0$.
 (b) Let $P \subset \mathcal{O}_K$ be a nonzero prime ideal. How many elements of $P \cap \mathbb{Z}$ are primes in $\mathbb{Z}$? Justify your answer. (3+3)
2. Let A be a Dedekind domain. Let I, J be two fractionary ideals of A . Prove that $IJ = (I \cap J)(I + J)$. (6)
3. Let $\omega \in \mathbb{C}$ be a primitive cube root of unity. Prove that (a) $1 - \omega$ is a prime in $\mathbb{Z}[\omega]$. (b) $3 = u(1 - \omega)^2$ for some unit u in $\mathbb{Z}[\omega]$. (c) Obtain a prime factorization of the ideal $\langle 3 \rangle$ of $\mathbb{Z}[\omega]$. (3+3+2)
4. Let $K = \mathbb{Q}(\sqrt{3})$ and $L = \mathbb{Q}(\sqrt{7})$. (a) Compute $[KL : \mathbb{Q}]$. (b) Give an integral basis for KL , with justification. (2+4)
5. Let $m \geq 3$ and let $\omega = e^{2\pi i/m}$. Compute $N_{K/\mathbb{Q}}(\omega)$. (6)
6. Let $A = \mathbb{Z}[\sqrt{-3}]$ and $I \subset A$ be the ideal $I = \langle 2, 1 + \sqrt{-3} \rangle$.
 (a) Show that $I \neq \langle 2 \rangle$ and $I^2 = 2I$. (b) Prove that the ideal $\langle 2, 1 + \sqrt{-3} \rangle$ is a maximal ideal in A . (c) Show that A is not a Dedekind domain. (2+2+2)
7. Let $K = \mathbb{Q}(\sqrt{-3})$. Find an integral basis for K and justify your answer. (6)
8. Is the ring $A = \mathbb{Z}[\sqrt{-5}]$ a UFD? Explain. Is A the ring of integers in $\mathbb{Q}(\sqrt{-5})$? Explain your answer. (3+3)